History of Geometry

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Getting Started :

In this lecture, I propose to put forward a historical account of developments in Geometry. The purpose of this lecture is to share the excitement of tracing the origins of fundamental notions in geometry and at the same time reviewing the progression of the human thought process. This is a guided tour to my dreamland and I hope you will enjoy the journey. With this kind of intelligent but heterogeneous audience, the choice for the topic of the presentation was not easy. Surely, I cannot bring in mathematical technicalities. So the mathematics audience should pardon me for not doing actual mathematics but at the same time non-mathematics audience should thank me for taking a decision in their favour. The hero of the story is of course the Greek Thinker, Philosopher and Mathematician : **Euclid of Alexandria** who lived around 300 B.C.

Before we get into the celebrated works of Euclid, let us take a quick review of geometry before Euclid .

Geometry Before Euclid :

The first geometrical considerations of man are very ancient and would seem to have their origins in simple observations stemming from human ability to recognize physical forms and to compare shapes and sizes. Many observations in the daily life of early man must have led to the conception of curves, surfaces and solids. Periphery of sun or moon, the rainbow, the shadows cast by the sun, seed heads of flowers are instances of circle. Many fruits and pebbles are spherical, bubbles in water are hemispherical, some bird eggs are ellipsoidal, tree trunks are cylindrical. Such examples of ordered forms, shapes and sizes occur abundantly in nature. These forms, shapes and sizes attracted the attention of a reflective mind in the primitive ages and thus some elementary geometrical concepts were brought to light. We may refer to these notions as ***subconscious geometry***. At this particular primitive stage in the history of human intellect, we do not see any evidence of realization of geometry as a scientific discipline but we do observe that the notion of symmetry was captured in the human brain. (**Subconscious geometry plays an important role in the development of an artist.)**

We take up this particular evolutionary stage as a background for Geometry which developed later as a profound science and its progress in diverse directions turned out to be extremely dramatic.

There is no evidence of the exact number of centuries that passed before man was able to raise geometry to the status of science. When human intellect was able to extract from a concrete relationship, a general abstract relationship containing the concrete as a special case, geometry-and for that matter any branch of knowledge achieved the status of science. One thus arrives at a notion of geometrical law. The Nile Valley of ancient Egypt is believed to be the place where the subconscious geometry first became **practical geometry**. Scientific geometry arose from practical necessity as a science to assist in engineering, agriculture, business and religious ritual. Civilization reached a high level in Egypt at an early period (approximately 2500 B.C.) . The roots of scientific geometry could be traced back to Egyptian surveying practices (geometry means " measurement of earth") and construction of pyramids . The great pyramid of Giza was built around 2600 B.C. In the Berlin museum, there are two instruments from ancient Egypt of about 2000 B. C. - an oldest surveying machine and a sundial. There are also evidences of early geometry in Mesopotamia - again a fertile land between the rivers Tigris and Euphrates. This civilization is normally referred to as Babylonian civilization. The Babylonian period reveals that particular cases of Pythagoras Theorem were known to them. The culmination of practical geometry into scientific geometry took many centuries. The earliest evidence of scientific geometry could be found in the works of Greek mathematicians Thales and Pythagoras. It is believed that both of them travelled to Egypt and Babylonia. Thales is regarded as the first philosopher and the first of the seven wise men. Thales had been a man of practical affairs but Pythagoras was a prophet and a mystic. He was a contemporary of Buddha, Confucius and Loa-Tze so that the century was critical in the development of religion as well as mathematics. He founded a philosophical and religious school in Croton around 535 B.C. The fourth century B.C. opened with the death of Socrates, a scholar who adopted the dialectic method. Plato a student and admirer of Socrates became a mathematical inspiration of 4^{th} century B.C. Plato founded an Academy in Athens and it was the centre of the mathematical activity at that time and guided and inspired its development. He instilled a mathematical culture very strongly in the society. Over the doors of his academy was inscribed a motto, " Let no one ignorant of geometry enter here". (425 B.C. – 340 B.C.) . Aristotle (- 322 B.C.) the most widely learned scholar of all times was a philosopher and a biologist. He insisted on logical presentation of knowledge and codified syllogisms.

In 323 B.C., Alexander the great suddenly died and his empire fell apart. The Egyptian portion of the empire was firmly in the hands of Ptolemy I , a very enlightened ruler who was able to turn his attention to constructive efforts. Among his early acts was the establishment of an institute at Alexandria, known as "**Museum**" . As teachers of the school, he invited leading scholars among whom was the author of the most fabulously successful mathematics textbook ever written – "**Elements**" of **Euclid** . The book was composed in about 300 B.C. and was copied and recopied repeatedly after that. Thousands of editions and versions of the book have been released and in terms of popularity and sale it can be compared only to the Bible. First printed copy appeared in 1482 at Venice. With the decline of great civilizations of Athens and Rome it moved eastward to the centre of Arabic learning. In the late middle ages it was translated from Arabic to

Latin and since Renaissance it not only has been the most widely used textbooks in the world but has had an influence as a model of scientific thought that extends way beyond the confines of geometry. Most of the results in Elements were known long before Euclid. Euclid's great contribution is that he organized the geometrical knowledge of his time into a coherent logical framework, whereby each result could be deduced from that preceding it, starting with only a small number of "postulates' regarded as self evident.

The most naïve approach to geometry is to regard it as a collection of facts or truths about the real world. Ancient geometry began as a set of useful rules for measuring fields, construction of pyramids, etc. During Euclid's time we can detect two important changes in the perception of geometry. One concerns with the nature of the geometrical truth. There is a distinction between all the imperfections of the real world and some kind of ideal, abstract existence that people in this world strive to attain. In fact this is a typical Platonic point of view. Thus geometry was elevated from the status of practical science to the study of relationships. Euclid's geometry is the geometry of this ideal world in the sense of Plato with emphasis on exact relationships.

Euclidean Geometry :

One of the striking features of Euclidean Geometry is its logical structure. Starting from a small set of definitions and postulates, all the succeeding results are proved by a logical deduction from what has gone before. The method is now widely known as Axiomatic Method. Without doubt, Euclid was the first to conceive and successfully exhibit the most powerful and logically sound technique of mathematical exposition. In order to appreciate Euclid's tremendous accuracy and vision and also to enjoy the exciting story of Non- Euclidean Geometry, we need to understand the axioms of the Euclidean geometry. These axioms are given below :

1. To draw a straight line passing through any two points.
2. To produce a finite straight line continuously into a straight line.
3. To describe a circle with any centre and radius.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then two straight lines, produced indefinitely, meet on that side on which are the angles less than two right angles.

The succeeding mathematicians had no difficulty in accepting the first four postulates. The fifth postulate turned out to be controversial ! The statement sounds more like a theorem than an axiom. It definitely lacks in the simple comprehensibility of the other four axioms. Euclid himself avoided the use of the fifth postulate until he comes to the proof of his 29^{th} proposition ! In view of the complicated nature of the fifth postulate, several attempts were made, over the centuries, to reformulate the postulate or to deduce it as a consequence of other postulates and the first 28 propositions of Euclid. But all these attempts proved unsuccessful !! In a way, all these attempts repeatedly proved the vision of the great master who very correctly thought of the statement to be an axiom. On the other hand, in the process of attacking the fifth postulate, a lot of interesting and beautiful mathematics came into existence. In attempting to prove the fifth postulate, several mathematicians came up with different statements equivalent to the fifth postulate. Some famous statements are given below :

1. John Playfair : Through a given point not on a given line only one line parallel to a given line can be drawn.
2. Posidonius and Geminus : There exist a pair of coplanar lines, everywhere equidistant from one another.
3. Wallis, Sacheri, Carnot and Laplace : There exists a pair of similar non- congruent triangles.
4. Sacheri : If in a quadrilateral, a pair of opposite sides are equal and if angles adjacent to the third side are right angles, then the other two angles are also right angles.
5. Lambert and Clairaut : If in a quadrilateral, three angles are right angles, the fourth angle is also a right angle.
6. Legendre : there exists at least one triangle having the sum of its three angles equal to two right angles.
7. Legendre : Through any point within an angle less than 60 degrees, one could always draw a straight line intersecting both sides of the angle.
8. Legendre and Bolyai : A circle can be passed through any three non-collinear points.
9. Gauss : there is no upper limit to the area of a triangle.

It is an interesting exercise in plane geometry to establish the equivalence of all these statements with the fifth postulate ! I would like to make an appeal to the mathematics readers to supply the proofs.

Non- Euclidean Geometry :

We now discuss the evolution of Non-Euclidean Geometry –a beautiful branch of mathematics developed through the efforts in proving the fifth postulate. The major contributions came from Sacheri (1667-1733), Lambert (1728-177), Legendre (1752- 1833), Johann Boyai (1802-1860), Lobatchevsky (1793-1856), Riemann (1826-1866). In what follows, we briefly discuss the contributions of these mathematicians :

1. **Sacheri** : He was the first to use the method of reductio absurdum to prove the fifth postulate. He started with a quadrilateral with two adjacent right angles. Then without using the fifth postulate it can be proved that the other two angles are equal (try !). Now if it can be

proved that these two equal angles are right angles, then the fifth postulate is proved ! So he assumed the contrary i. e. these two angles are either both acute or both obtuse and then kept hoping for a contradiction. But the exciting part of the story is that he could never get any contradiction !! In the process he proved some results under the acute and obtuse angle hypothesis. Well ! As we can now say that these are the first few theorems of the Non-Euclidean Geometry ! Of course Sacheri, though, did not dare to call them as theorems but in fact somehow very weakly rounded his arguments in an unconvincing manner. Hence despite substantial contributions, he could not get the credit of discovering Non-Euclidean Geometry

2. **Lambert** : He started with a quadrilateral with three right angles and attempted to prove that the remaining angle is also a right angle but of course could not do that. In the process he proved that the defect or excess of a triangle is proportional to the area of the triangle. He also proved that in the geometry under the acute or obtuse angle hypothesis, the angle as well as lengths are absolute entities !
3. **Legendre** : He is known as the most prolific mathematician who along with other important contributions, substantially contributed to the development of Non-Euclidean Geometry. His various attempts could be found in his book ' Elements of Geometry', twelve editions of which were published during 1794-1823. He played a key role in popularizing the drama of the fifth postulate.
4. **Johann Bolyai** : He was the first to suspect the existence of a consistent Non-Euclidean geometry under the acute angle hypothesis. Along with Lobatchevsky he is known as the father of Non-Euclidean Geometry. His father Wolfgang was also a mathematician and also attempted the proof of the fifth postulate. In fact he advised his son not to work in that area . He said, " Don't traverse this bottomless night, which extinguished all light of my life". But of course the son never gave up ! He said, " Out of nothing I have constructed a strange new world ".
5. **Lobatchevsky** : Independent of the publications of Bolyai, he worked on Non-Euclidean Geometry and systematically presented Hyperbolic Non-Euclidean Geometry. His book ' Geometrical Researches in Theory of Parallels' is a milestone in this branch of mathematics.
6. **Riemann** : We all know that Riemann is one of the all time greats of mathematics and has done pioneering work in many branches of mathematics. He contributed substantially here too ! He developed the obtuse angle hypothesis and constructed the Riemannian Geometry. These discoveries later turned out to be useful in Einstein's Theory of Relativity.

   In the year 1868, Italian mathematician Blaterni, constructed the models for both Hyperbolic and Riemannian Geometries and thus proved the consistency of these systems. This finally settles the problem of the fifth postulate !

Philosophical Consequences :

Some consequences of the consistency of Non-Euclidean Geometries are much more far reaching than the settlement of the fifth postulate. As a result, it was revealed that the truth or falsity of a geometrical system is not a matter of concern but any consistent system is theoretically as good as Euclid's system! One may have a different set of geometries for different applications. This point of view is in sharp contrast to the Kantian theory of space which is based on the assumption of a given, intuitive concept of space !

Well ! Looking at all these dramatic incidents one is simply amazed by the vision, accuracy, beauty consistency, ... that is visible on every page of Euclid's wonderful book 'Elements' ! I can't resist quoting Bertrand Russell :

"At the age of eleven I began Euclid ... This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in this world "

References :

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